Justification logic for constructive modal logic *

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Abstract

We provide a treatement of the intuitionistic $\Diamond$ modality in the style of justification logic. We introduce a new type of terms, called witness terms, that justify consistency, obtain justification analogs for the constructive modal logics CK, CD, CT, and CS4, and prove the realization theorem for them.

1 Introduction

Justification logic is a family of modal logics generalizing the Logic of Proofs LP, introduced by Artemov in [2]. The original motivation, which was inspired by works of Kolmogorov and Gödel in the 1930's, was to give a classical provability semantics to intuitionistic propositional logic. Gödel [15] made the first steps by translating intuitionistic logic into the modal logic S4, which he rediscovered as a logic of abstract provability. He noted that S4-provability is incompatible with arithmetical reasoning due to the former's acceptance of the reflection principle and outlined, in an unpublished lecture [16], a potential way of overcoming this obstacle by descending to the level of proofs rather than provability. Artemov independently implemented essentially the same idea in the Logic of Proofs by showing that it provides an operational view of the same type of provability as S4 [2, 3].

The language of the Logic of Proofs can be seen as a modal language where occurrences of the $\Box$-modality are replaced with terms, also known as *proof polynomials*, *evidence terms*, or *justification terms*, depending on the setting. The intended meaning of the formula '$t:A$' is '*t is a proof of A*' or, more generally, the reason for the validity of A. Thus, the justification language is viewed as a refinement of the modal language, with one provability construct $\Box$ replaced with an infinite family of specific proofs.

It gradually became clear that the applicability of this result goes way beyond the provability interpretation of the modality, and can be equally well considered in other settings, including, notably, epistemic logic [5]. Indeed, the connection between the Logic of Proofs and the modal logic S4 has been extended to other modal logics (based on classical propositional reasoning), including normal modal sublogics of S4 [10], the modal logic S5 [8], all 15 logics of the so-called modal cube between the minimal normal modal logic K and S5 [17], the infinite family of Geach logics [14], etc.

The correspondence between a justification logic and a modal logic means that erasing specific reasons in a valid statement about proofs leads to a valid statement about provability and, vice versa, any valid statement about provability can be viewed as a *forgetful projection* of a valid statement about proofs. Moreover, this existential view of $\Box$ as '*there exists a proof*' leads to a first-order provability reading of modal statements and suggests that they can be Skolemized. Such a Skolemization makes negative occurrences of $\Box$ into Skolem variables and positive occurrences into Skolem functions, suggesting a further restriction on the way the $\Box$ modalities are filled in with proof terms—the process called *realization*—negative occurrences should be filled in with distinct proof variables.

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The Logic of Proofs was born out of an analysis of intuitionistic logic with the goal of explaining it using *classical reasoning* about proofs. However, other relationships with intuitionistic logic have also been explored. Artemov introduced the first intuitionistic version ILP of the Logic of Proofs in [4] to unify the semantics of modalities and lambda-calculus. Indeed, as simply typed lambda-calculus is in correspondence with intuitionistic proofs, he needed to define an intuitionistic axiomatization of the Logic of Proofs to relate modal logic S4 and lambda-calculus. His axiomatisation simply changes the propositional base to intuitionistic while keeping the other axioms of Logic of Proofs unchanged. He shows that ILP is in correspondence with the $\Box$-only fragment of the constructive logic CS4 as defined in [9].[1] Recently, Marti and Studer [19] supplied ILP with possible worlds semantics akin to the semantics developed by Fitting for the classical Logic of Proofs [13].

However, this axiomatisation is not enough to obtain a proper intuitionistic arithmetical semantics, that is, to interpret '$t : A$' as '*t is a proof of A in Heyting Arithmetic*', which is the motivation behind another line of work for considering intuitionistic versions of the Logic of Proofs. In order to obtain an intuitionistic Logic of Proofs complete for Heyting arithmetic, Iemhoff and Artemov [7] added to ILP extra axioms that internalize admissible rules of intuitionistic propositional logic. The arithmetical completeness was later shown by Dashkov [**?**]. Finally, Steren and Bonelli [22] provide an alternative system of terms for ILP based on natural deduction with hypothetical judgements.

What unifies all these versions of intuitionistic justification logics is the exclusive attention to the provability modality. Be the focus on semantics, realization theorem, or arithmetical completeness, the modal language is restricted to the $\Box$ modality. This restriction was quite natural in the classical setting, where $\Diamond$ can simply be viewed as the dual of $\Box$. However, with the freedom of De Morgan shackled comes the responsibility to treat $\Diamond$ as a fully independent modality—a responsibility that we take upon ourselves in this paper. In this first exploration of the kind of terms necessary to represent the operational side of the intuitionistic $\Diamond$ modality, we concentrate on *constructive versions* of several modal logics.

Building on Artemov's treatment of the $\Box$-only fragment, we add a second type of terms, which we call *witness terms* and denote by Greek letters. Thus, a formula $\Diamond A$ is to be realized by '$\mu : A$'. The intuitive understanding of these terms is based on the view of $\Diamond$ modality as representing consistency (with $\Box$ still read as provability). The term μ justifying the consistency of a formula is viewed as an abstract witnessing model for the formula. We keep these witnesses abstract so as not to rely on any specific semantics. All the operations on witness terms that we employ to ensure the realization theorem for CK, CD, CT, and CS4, as defined in [26, 20, 9], are akin to the operations on proof terms. In particular, the operation $+$ for proof concatenation finds a counterpart in the operation $\sqcup$ for disjoint model union. Similarly, the application operation $\cdot$ that internalizes reasoning by modus ponens (if t is a proof of $A \supset B$ and s is a proof of A, then $t \cdot s$ is a proof of B) has a counterpart $\star$ that creates a witness for B from a proof of $A \supset B$ and a witness for A. The intuition behind the *witness execution* operation $\star$ is that a proof of $A \supset B$, when applied to a witness for A provides evidence that the same model is also a witness for B.

Outline of the paper: In Sect. 2, we introduce the syntax and proof theory of some constructive modal logics, and in Sect. 3, we give our definition of a justification logic for constructive modal logics. Then, in Sect. 4, we prove the main theorem of this paper, the realization theorem linking the various constructive modal logics to the corresponding justification logic. Finally, in Sect. 5, we point to further questions left as future work, as this paper is only the beginning of the research program consisting in giving justification logic for constructive and intuitionistic versions of modal logics.

[1] Artemov himself calls the logic CS4 "the intuitionistic modal logic on the basis of S4" and denotes it IS4.

$$\mathsf{k}_1\colon\ \Box(A\supset B)\supset(\Box A\supset\Box B) \qquad \mathsf{d}\colon\ \Box A\supset\Diamond A$$
$$\mathsf{k}_2\colon\ \Box(A\supset B)\supset(\Diamond A\supset\Diamond B) \qquad \mathsf{t}\colon\ (A\supset\Diamond A)\wedge(\Box A\supset A)$$
$$\mathsf{4}\colon\ (\Diamond\Diamond A\supset\Diamond A)\wedge(\Box A\supset\Box\Box A)$$

Figure 1: Modal axioms used in this paper

2 Constructive modal logic

Let $a\in\mathscr{A}$ for a countable set of propositional variables $\mathscr{A}=\{a,b,c,\dots\}$. We define the grammar:

$$A ::= \bot \mid a \mid A\wedge A \mid A\vee A \mid A\supset A \mid \Box A \mid \Diamond A \tag{1}$$

We use capital Latin letters (A, B, C, ...) for formulas and define the negation as $\neg A := A\supset\bot$.

In modal logic the behavior of the $\Box$-modality is determined by the k-axiom $\Box(A\supset B)\supset\Box A\supset\Box B$ and by the *necessitation rule* saying that if A is valid then so is $\Box A$, be the logic classical or intuitionistic. In classical modal logic the behavior of the $\Diamond$-modality is then fully determined by the De Morgan duality, which is violated in the intuitionistic case. This means that more axioms are needed to define the behavior of the $\Diamond$.

However, there is no unique way of doing so, and consequently many different variants of "intuitionistic modal logic" do exist. In this paper we consider the variant that is now called *constructive modal logic* [26, 9, 20, 1] and that is defined by adding to intuitionistic propositional logic the two axiom schemes shown in the left column of Fig. 1 together with the necessitation rule mentioned above. We call this logic CK. We also consider (i) the logic CD, which is CK extended with the d-axiom, (ii) the logic CT which is CK extended with the t-axiom , and (iii) the logic CS4 which is CT extended with the 4-axiom, all three axioms shown in the right column of Fig. 1.

These four logics have simple sequent calculi that can be obtained from any sequent calculus of intuitionistic propositional logic by adding the appropriate rules for the modalities. In this paper, a *sequent* is an expression of the shape $B_1,\dots,B_n\Rightarrow C$ where $B_1,\dots,B_n,C$ are formulas and $B_1,\dots,B_n$ has to be read as multiset (i.e., the order is irrelevant, but it matters how often a formula appears). We use capital Greek letters (Γ, Δ, Σ, ...) to denote such multisets of formulas. For a sequent $B_1,\dots,B_n\Rightarrow C$ we define its *corresponding formula* as $fm(B_1,\dots,B_n\Rightarrow C) := B_1\wedge\cdots\wedge B_n\supset C$.

We start from the standard sequent calculus G3ip [24] whose rules are shown in Fig. 2. Then, the systems for the logics CK, CD, CT, and CS4, that we call LCK, LCD, LCT, and LCS4 respectively, are obtained by adding the rules in Fig. 3 according to the following table.[2]

$$\begin{aligned}\mathsf{LCK} &= \mathsf{G3ip}+\mathsf{k}_\Box+\mathsf{k}_\Diamond\\ \mathsf{LCD} &= \mathsf{G3ip}+\mathsf{k}_\Box+\mathsf{k}_\Diamond+\mathsf{d}\\ \mathsf{LCT} &= \mathsf{G3ip}+\mathsf{k}_\Box+\mathsf{k}_\Diamond+\mathsf{t}_\Box+\mathsf{t}_\Diamond\\ \mathsf{LCS4} &= \mathsf{G3ip}+\mathsf{4}_\Box+\mathsf{4}_\Diamond+\mathsf{t}_\Box+\mathsf{t}_\Diamond\end{aligned} \tag{2}$$

Observe that the axiom rule id is restricted to atomic formulas. We rely on that in the proof of the realization theorem in Sect. 4. But as expected, using the standard argument by induction on the formula construction, the general form of the axiom rule is derivable

Lemma 2.1 (Generalized axioms)**.** *For every formula A, the rule* $\mathsf{id}_g\ \dfrac{}{\Gamma,A\Rightarrow A}$ *is derivable in* G3ip*,* LCK*,* LCD*,* LCT*, and* LCS4*.*

[2] For a survey of the classical variants of these systems, see, for example, [25].

$$\mathsf{id}\ \frac{}{\Gamma, a \Rightarrow a} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma, \bot \Rightarrow C}$$

$$\vee_{\mathsf{L}}\ \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\wedge_{\mathsf{L}}\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\supset_{\mathsf{L}}\ \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad \supset_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Figure 2: Sequent calculus G3ip for intuitionistic propositional logic

$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad \mathsf{k}_{\Diamond}\ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A} \qquad \mathsf{t}_{\Box}\ \frac{\Gamma, \Box A, A \Rightarrow B}{\Gamma, \Box A \Rightarrow B} \qquad \mathsf{t}_{\Diamond}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

$$\mathsf{d}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Diamond A} \qquad 4_{\Box}\ \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad 4_{\Diamond}\ \frac{\Box\Gamma, B \Rightarrow \Diamond A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A}$$

Figure 3: Additional rules for modalities

Finally, the cut rule is given by $\mathsf{cut}\ \dfrac{\Gamma \Rightarrow A \qquad \Delta, A \Rightarrow C}{\Gamma, \Delta \Rightarrow C}$, and we have the following theorem:

Theorem 2.2 (Cut Admissibility)**.** *Let* L *be one of the systems* LCK*,* LCD*,* LCT*, or* LCS4*. If a sequent is provable in* $\mathsf{L} + \mathsf{cut}$ *then it is also provable in* L*.*

For LCK, LCD, and LCT, the proof follows as a special case from the work in [18], and for CS4 the result is stated in [9] as a "routine adaptation of Gentzen's method".

With the help of Theorem 2.2, we can easily show the completeness of our system:

Theorem 2.3 (Completeness)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$*, and let* LML *be the corresponding sequent system* (LCK*,* LCD*,* LCT*, or* LCS4 *respectively*)*. If* $\vdash_{\mathsf{ML}} A$*, then* $\vdash_{\mathsf{LML}} \Rightarrow A$*.*

Proof. The axioms of IPL can be proved using G3ip in Fig. 2; those in Fig. 1 can be proved using the corresponding rules in Fig. 3. Finally, the necessitation rule can be simulated with $\mathsf{k}_{\Box}$, and modus ponens can be simulated using cut. Now completeness of the cut-free systems follows immediately from Th. 2.2. □

Theorem 2.4 (Soundness)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$*. If* $B_1, \ldots, B_n \Rightarrow C$ *is a sequent provable in the corresponding sequent system* LML*, then the corresponding formula* $B_1 \wedge \cdots \wedge B_n \supset C$ *is a theorem of* ML*.*

Proof. We proceed by induction on the proof π in LML, making a case analysis on the bottom-most rule instance in π. For the rules in G3ip, this is straightforward. Now consider the rule

$$\mathsf{k}_{\Box}\ \frac{C_1, \ldots, C_n \Rightarrow A}{\Box C_1, \ldots, \Box C_n, D_1, \ldots, D_m \Rightarrow \Box A}$$

$$\begin{array}{rl}
\mathsf{taut}: & \text{Complete finite set of axioms for intuitionistic propositional logic} \\
\mathsf{jk}_{\Box}: & t:(A\supset B)\supset(s:A\supset t\cdot s:B) \\
\mathsf{jk}_{\Diamond}: & t:(A\supset B)\supset(\mu:A\supset t\star\mu:B) \\
\mathsf{sum}: & s:A\supset(s+t):A \text{ and } t:A\supset(s+t):A \\
\mathsf{union}: & \mu:A\supset(\mu\sqcup\nu):A \text{ and } \nu:A\supset(\mu\sqcup\nu):A
\end{array}$$

$$\mathsf{mp}\ \frac{A\supset B \quad B}{B} \qquad \mathsf{ian}\ \frac{A \text{ is an axiom instance}}{c_1:\ldots c_n:A}$$

Figure 4: Axiomatization of the constructive justification logic JCK

$$\begin{array}{llll}
\mathsf{jd}_{\Box}: & t:\bot\supset\bot & \mathsf{jd}_{\Diamond}: & \top\supset\mu:\top \\
\mathsf{jt}_{\Box}: & t:A\supset A & \mathsf{jt}_{\Diamond}: & A\supset\mu:A \\
\mathsf{j4}_{\Box}: & t:A\supset !t:t:A & \mathsf{j4}_{\Diamond}: & \mu:\nu:A\supset\nu:A
\end{array}$$

Figure 5: Additional justification axioms

By induction hypothesis we have that $\vdash_{\mathsf{ML}} C_1\wedge\cdots\wedge C_n\supset A$, and therefore by intuitionistic reasoning also that $\vdash_{\mathsf{ML}} C_1\supset\cdots\supset C_n\supset A$.[3] By necessitation we obtain that $\vdash_{\mathsf{ML}}\Box(C_1\supset\cdots\supset C_n\supset A)$, and using k_1 and modus ponens we get $\vdash_{\mathsf{ML}}\Box C_1\supset\cdots\supset\Box C_n\supset\Box A$. Then, $\vdash_{\mathsf{ML}}\Box C_1\wedge\cdots\wedge\Box C_n\wedge D_1\wedge\cdots\wedge D_m\supset\Box A$ follows by intuitionistic reasoning. The other cases are similar. □

3 Justification logic

Justification logic adds *proof terms* directly inside its language using formulas '$t:A$' with the meaning '*t is a proof of A*'. In the constructive version that we propose in this section, we will also add *witness terms* into the language, using formulas '$\mu:A$' with the underlying intuition that '*μ is a model of A*'.

Proof terms, intended to replace $\Box$, are denoted $t,s,\ldots$, while *witness terms*, intended to replace $\Diamond$, are denoted $\mu,\nu,\ldots$ Proof terms are built from a set of *proof variables*, denoted $x,y,\ldots$, and a set of *(proof) constants*, using the following operations: $\cdot$ *application*, $+$ *sum*, and $!$ *proof checker*. Witness terms are built from a set of *witness variables*, denoted $\alpha,\beta,\ldots$, using the operations $\sqcup$ *disjoint witness union* (combining two witness terms) and $\star$ *execution* (that combines a proof term with a witness term).

$$\begin{array}{rcccccccc}
t ::= & c & | & x & | & (t\cdot t) & | & (t+t) & | & !t \\
\mu ::= & & & \alpha & | & t\star\mu & | & (\mu\sqcup\mu) & &
\end{array}$$

The formulas of justification logic are then obtained from the following grammar:

$$A ::= \bot \mid a \mid (A\wedge A) \mid (A\vee A) \mid (A\supset A) \mid t:A \mid \mu:A$$

We propose to extend the definition of justification logics to realize constructive modal logics. The axiomatization of the basic one is shown in Fig. 4. It is similar to the standard justification counterpart of the classical modal logic K except for the additional axiom $\mathsf{jk}_{\Diamond}$ which corresponds to the modal axiom k_2. The other axioms taut, $\mathsf{jk}_{\Box}$, sum, and the rules of *modus ponens* mp and *iterated axiom necessitation* ian are standard, e.g. see [17]. We call this basic logic JCK, and as in the classical setting, we can define

[3] Throughout the paper we consider $\supset$ to be right-associative.

extension of JCK using the axioms defined in Fig. 5. The logic JCD is obtained from JCK by adding the axioms $\mathsf{jd}_\Box$ and $\mathsf{jd}_\Diamond$; the logic JCT is obtained from JCK by adding the axioms $\mathsf{jt}_\Box$ and $\mathsf{jt}_\Diamond$; and the logic JCS4 is obtained by adding the axioms $\mathsf{j4}_\Box$ and $\mathsf{j4}_\Diamond$ to JCT. Note that the $\Box$-variants of each axiom corresponds exactly to the one used in the classical setting. Our contribution is the definition of the $\Diamond$-variants operating on the witness terms instead of the proof terms.

The logics JCK, JCD, JCT, and JCS4 can be seen as the operational version of the constructive modal logics CK, CD, CT, and CS4, defined in the previous section. Indeed if one forgets about the proof and witness terms annotations and considers them as empty $\Box$ and $\Diamond$ respectively, the logics prove the same theorems.

Definition 3.1. We define the operation of *forgetful projection* $(\cdot)^\circ$ that maps justification formulas onto corresponding modal formulas by recursion on the construction of justification formulas:

$$\begin{array}{ll} \bot^\circ := \bot & \qquad (t:A)^\circ := \Box A^\circ \\ a^\circ := a \quad \text{if } a \text{ is a propositional variable} & \qquad (\mu:A)^\circ := \Diamond A^\circ \\ & \qquad (A * B)^\circ := A^\circ * B^\circ \quad \text{for } * \in \{\wedge, \vee, \supset\} \end{array}$$

We extend this definition to multisets of formulas: $(A_1, \ldots, A_n)^\circ := A_1^\circ, \ldots, A_n^\circ$.

The following lemma is easy to show by induction on the Hilbert derivation in a justification logic JL:

Lemma 3.2 (Forgetful projection)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$ *and* ML *the corresponding modal logic. If* $\vdash_{\mathsf{JL}} F$*, then* $\vdash_{\mathsf{ML}} F^\circ$.

The more difficult question however is: can we get the converse? This result is called realization, namely that every theorem of a certain modal logic can be '*realized*' by a justification theorem. However, it is not such an easy result as it may seem. It is not possible to transform directly a Hilbert proof of a modal theorem into a Hilbert proof of its realization in justification logic as the rule mp in a Hilbert system can create dependencies between modalities. The standard solution to this issue is to consider a proof of the modal theorem in a *cut-free* sequent calculus as the absence of cuts in the proof will prevent the creation of dependencies. The detailed statement and proof of this result can only be presented in the next section, as we have to introduce some basics first.

We state below two lemmas that are crucial for the realization proof: the *Lifting Lemma* and the *Substitution Property*. They are extensions of standard results from the justification logics literature to the constructive case. Repeating verbatim the proof from [3], we obtain the *Lifting Lemma* and its variant showing that necessitation can be internalized within the language of each of these justification logics.

Lemma 3.3 (Lifting Lemma)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If* $A_1, \ldots, A_n \vdash_{\mathsf{JL}} B$*, then there exists a proof term* $t(x_1, \ldots, x_n)$ *such that* $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B$ *for all terms* $s_1, \ldots, s_n$.

Corollary 3.4. *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If* $\vdash_{\mathsf{JL}} A_1 \wedge \ldots \wedge A_n \supset B$*, then there exists a proof term* $t(x_1, \ldots, x_n)$ *such that* $\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : B$ *for all terms* $s_1, \ldots, s_n$.

Furthermore, in our constructive setting, we will also need a $\Diamond$ variant of this statement.

Corollary 3.5. *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If* $\vdash_{\mathsf{JL}} A_1 \wedge \ldots \wedge A_n \wedge C \supset B$*, then there is a witness term* $\mu(x_1, \ldots, x_n, \beta)$ *such that* $\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B$ *for all terms* $s_1, \ldots, s_n, \nu$.

Proof. By intuitionistic reasoning and Corollary 3.4, we obtain a proof term $t(x_1, \ldots, x_n)$ such that

$$\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : (C \supset B).$$

Using the instance $t(s_1, \ldots, s_n) : (C \supset B) \supset \nu : C \supset (t(s_1, \ldots, s_n) \star \nu) : B$ of the axiom $\mathsf{jk}_\Diamond$, we can see that

$$\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B$$

for $\mu(x_1, \ldots, x_n, \beta) := t(x_1, \ldots, x_n) \star \beta$. □

Finally, we generalize the standard definition of substitution to our setting.

Definition 3.6. A *substitution* σ maps proof variables to proof terms and witness variables to witness terms. The application of a substitution σ to a term t or μ, denoted as $t\sigma$ or $\mu\sigma$, is defined inductively as follows:

$$\begin{array}{rr} c\sigma := c & x\sigma := \sigma(x) \\ (t \cdot s)\sigma := t\sigma \cdot s\sigma & (t+s)\sigma := t\sigma + s\sigma \\ (!t)\sigma := !(t\sigma) & \alpha\sigma := \sigma(\alpha) \\ (t \star \mu)\sigma := t\sigma \star \mu\sigma & (\mu \sqcup \nu)\sigma := \mu\sigma \sqcup \nu\sigma \end{array}$$

where c is a proof constant, x is a proof variable, and α is a witness variable. The application of σ to a justification formula A yields the formula $A\sigma$, where each term t (resp. μ) appearing in A is replaced with $t\sigma$ (resp. $\mu\sigma$).

The standard proof of the Substitution Property from [3] is also easily adaptable to our case.

Lemma 3.7 (Substitution Property)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$. *If* $\vdash_{\mathsf{JL}} A$*, then* $\vdash_{\mathsf{JL}} A\sigma$ *for any substitution* σ.

4 Realization theorem for constructive modal logic

Assume we have a justification formula F and its forgetful projection F°. In that case we call F a *realization* of F°. Similarly, a sequent $\Gamma \Rightarrow C$ is a *realization* of $\Gamma^\circ \Rightarrow C^\circ$. In order to define the notion of *normal realization* we need the notions of positive and negative occurrence of subformulas.

An occurrence of a subformula A of F is said to be *positive* if this position of A in the syntactic tree of F is reached from the root by following the left branch of a $\supset$ branching an even number of times; otherwise it is said to be *negative*. For example, the subformula A in the formula $(A \supset B) \supset C$ is positive, while the subformula A in the formula $A \supset (B \supset C)$ is negative. The *polarity* of the occurrence of a subformula in a sequent $\Gamma \Rightarrow C$ is given by its polarity in the formula $fm(\Gamma \Rightarrow C)$.

Definition 4.1. A realization $\Gamma \Rightarrow C$ of $\Gamma^\circ \Rightarrow C^\circ$ is called *normal* if the following conditions are fulfilled:

- if $t : A$ is a negative subformula of $\Gamma \Rightarrow C$ then t is a proof variable,
- if $\mu : A$ is a negative subformula of $\Gamma \Rightarrow C$ then μ is a witness variable, and
- all these variables are pairwise distinct.

We can now state and prove the main theorem of this paper.

Theorem 4.2 (Realization)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$*, let* JL *be the corresponding justification logic, i.e.,* JCK*,* JCD*,* JCT*, or* JCS4 *respectively, and let* LML *be the cut-free sequent calculus for* ML*. If* $\vdash_{\mathsf{LML}} \Gamma' \Rightarrow C'$ *for a given sequent of modal formulas, then there is a normal realization* $\Gamma \Rightarrow C$ *of* $\Gamma' \Rightarrow C'$ *such that* $\vdash_{\mathsf{JL}} fm(\Gamma \Rightarrow C)$.

Corollary 4.3. *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$ *and let* JL *be* JCK*,* JCD*,* JCT*, or* JCS4 *respectively. If* $\vdash_{\mathsf{ML}} A$*, then there is a justification formula* F *such that* $F^\circ = A$ *and* $\vdash_{\mathsf{JL}} F$.

Proof of Theorem 4.2. The proof goes largely along the lines of that for the $\Box$-only classical fragment (see [3, 10]). The operation $\sqcup$ on witness terms plays the same role as the operation $+$ on proof terms. Thus, we only show in detail cases for the new rules.

Let π be the LML proof of $\Gamma \Rightarrow C$. Let n be the number of $\Box$ and $\Diamond$ occurrences in the endsequent $\Gamma \Rightarrow C$. We assign to each of these $\Box$ and $\Diamond$ occurrences a unique index $i \in \{1, \ldots, n\}$.

Let us define the *modal flow graph* of π, denoted G_π, as follows: its vertices are all occurrences of formulas of the form $\Box A$ and $\Diamond A$ in π. Two such occurrences are connected with an edge iff they are occurrences of the same formula and either

- one occurs within a side formula in the premise of the rule and the other is the same occurrence within the same subformula in the conclusion of a rule or
- one occurs within an active formula in the premise of the rule and the other is the corresponding occurrence within the principal formula in the conclusion of the rule.

Each connected component of G_π has exactly one vertex in the endsequent of π and all vertices in the connected component are assigned the same index as this representative in the endsequent. E.g., in the following instance of $\mathsf{k}_\Box$. modalities connected by edges are vertically aligned and given the same index:

$$\mathsf{k}_\Box \frac{\Box_5 a \vee \Diamond_7 b \ , \qquad c \supset d \qquad\qquad \Rightarrow \qquad \Diamond_9 g \supset \Box_8 h}{\Box_2(\Box_5 a \vee \Diamond_7 b), \Box_6(c \supset d), \Box_3 e, \Box_{10}\Diamond_{20} f \Rightarrow \Box_{15}(\Diamond_9 g \supset \Box_8 h)} \tag{3}$$

The resulting graph is a forest where each tree has its root in the endsequent and is identified with a unique modality type $\heartsuit$ and unique index i. We denote such a tree a $\heartsuit_i$*-tree*. Branching occurs in the branching rules, as well as in the rules with embedded contraction, e.g., in $\mathsf{t}_\Box$ each modality in A within $\Box A$ in the conclusion of the rule branches to the corresponding occurrence in A and the corresponding occurrence in $\Box A$ in the premise. Leaves of the trees occur either in side formulas of the axioms id or $\bot_\mathsf{L}$, we call them *initial leaves*, or in the conclusions of the modal rules in Figure 3 when the modality with this index is not present in the premise of the rule, we call these *modal leaves*. For instance, the $\Box_2$-, $\Box_6$-, $\Box_3$-, $\Box_{10}$-, $\Diamond_{20}$-, and $\Box_{15}$-trees have leaves in the conclusion of (3).

We call the number of modal leaves of the $\heartsuit_i$-tree occurring in the succedents of the corresponding modal rules the *multiplicity of* i, denoted by m_i, which is a non-negative integer.

From the tree π, whose vertices are modal sequents, we construct the tree π_0 whose vertices are justification sequents by replacing

each $\Box_i$ s.t. $m_i > 0$ with the proof term $z_i := (y_{i,1} + \cdots + y_{i,m_i})$ for proof variables $y_{i,1}, \ldots, y_{i,m_i}$;

each $\Box_i$ s.t. $m_i = 0$ with the proof term $z_i := y_{i,0}$ for a proof variable $y_{i,0}$;

each $\Diamond_i$ s.t. $m_i > 0$ with the witness term $\omega_i := (\beta_{i,1} \sqcup \cdots \sqcup \beta_{i,m_i})$ for witness variables $\beta_{i,1}, \ldots, \beta_{i,m_i}$; and

each $\Diamond_i$ s.t $m_i = 0$ with the witness term $\omega_i := \beta_{i,0}$ for a witness variable $\beta_{i,0}$.

These variables are chosen in such a way that $y_{i_1,k_1} \neq y_{i_2,k_2}$ and $\beta_{i_1,k_1} \neq \beta_{i_2,k_2}$ whenever $(i_1,k_1) \neq (i_2,k_2)$.

Let us call a rule *justificational* if it is one of $\mathsf{k}_\Box$, $\mathsf{k}_\Diamond$, d, or $\mathsf{4}_\Box$ or *simple* otherwise (in particular, the rules $\mathsf{4}_\Diamond$, $\mathsf{t}_\Box$, $\mathsf{t}_\Diamond$ and all the rules in Figure 2 are simple). Let k be the number of instances of justificational rules in π. We construct a sequence of substitutions $\sigma_1, \ldots, \sigma_k$ that, applied to π_0 produces a sequence $\pi_0, \ldots, \pi_k$ of trees such that $\pi_{h+1} = \pi_h \sigma_{h+1}$. Note that for any sequent $\Delta \Rightarrow D$ in a tree π_h, its forgetful projection $\Delta^\circ \Rightarrow D^\circ$ is the modal sequent from the corresponding node of the tree π and that every occurrence of $\Box_i$ or $\Diamond_i$ in π is replaced in π_h with $z_i\sigma_1 \ldots \sigma_h$ or $\omega_i\sigma_1 \ldots \sigma_h$ respectfully. Let us denote $\tau_h := \sigma_h \circ \cdots \circ \sigma_1$ and call $\Delta \Rightarrow D$, $z_i\tau_h$, and $\omega_i\tau_h$ the *h-prerealizations* of $\Delta^\circ \Rightarrow D^\circ$, $\Box_i$, and $\Diamond_i$ respectively.

Let the k justificational rules be ordered linearly in a way consistent with the tree order of π. In other words, for arbitrary $k \geq j > i \geq 1$, the jth rule is not inside a subtree rooted at the premise of the ith rule.

Now we proceed by induction on $i = 0, \ldots, k$ to show that,

1. if none of the modal rules $i+1, \ldots, k$ are present in the subtree rooted at a sequent $\Delta \Rightarrow D$ in the tree π_i, then $\vdash_{\mathsf{JL}} fm(\Delta \Rightarrow D)$, i.e., the corresponding formula of the h-prerealization of $\Delta^\circ \Rightarrow D^\circ$ from π is derivable in JL provided all the justificational rules above an occurrence of this modal sequent in π are already processed.

2. $y_{i,0}\tau_h = y_{i,0}$ and $\beta_{i,0}\tau_h = \beta_{i,0}$, i.e., terms prerealizing modalities that are never introduced in the consequent of a justificational rule remain fixed points for all substitutions.

In particular, after all justificational rules are processed in π_k, the k-prerealization of the endsequent $\Gamma \Rightarrow C$ is derivable in JL making it a realization. Moreover, since no negative occurrence of a modality from the endsequent can be traced to a leaf in a right-hand side of a sequent π, in this realization all such negative modalities are realized by proof and witness variables. We prove it by a secondary induction on the depth of the proof up to the first unprocessed justificational rule.

For any simple rule, the JL-derivability of the premise(s) of the rule implies the JL-derivability of its conclusion. This fact is easy to prove by standard intuitionistic reasoning for the propositional rules of Fig. 2. In particular, for rules id and $\bot_{\mathsf{L}}$, we have $\vdash_{\mathsf{JL}} \bigwedge \Delta \wedge a \supset a$ and $\vdash_{\mathsf{JL}} \bigwedge \Delta \wedge \bot \supset D$. For $\mathsf{t}_\Box$ and $\mathsf{t}_\Diamond$, it follows from the axiom $t : A \supset A$ and axiom $A \supset \mu : A$ respectively. We provide the argument for

$$\mathsf{4}_\Diamond \frac{\Box_{k_1} G_1^\circ, \ldots, \Box_{k_r} G_r^\circ, B^\circ \Rightarrow \Diamond_j A^\circ}{\Box_{k_1} G_1^\circ, \ldots, \Box_{k_r} G_r^\circ, D_1^\circ, \ldots, D_p^\circ, \Diamond_l B^\circ \Rightarrow \Diamond_j A^\circ}$$

Assume that

$$\vdash_{\mathsf{JL}} \bigwedge_{r=1}^{m} z_{k_r} : G_r \wedge B \supset \omega_j \tau_h : A$$

for π_h. By Corollary 3.5, it follows that there exists a witness term μ such that

$$\vdash_{\mathsf{JL}} \bigwedge_{r=1}^{m} !z_{k_r} : z_{k_r} : G_r \wedge \omega_l : B \supset \mu(!z_{k_1}, \ldots, !z_{k_m}, \omega_l) : \omega_j \tau_h : A \ .$$

It now follows by $\mathsf{j4}_\Box$, $\mathsf{j4}_\Diamond$, and intuitionistic reasoning that

$$\vdash_{\mathsf{JL}} \bigwedge_{r=1}^{m} z_{k_r} : G_r \wedge \bigwedge_{n=1}^{p} D_n \wedge \omega_l : B \supset \omega_j \tau_h : A \ .$$

This observation alone establishes the base of the main induction, i.e., that all 0-prerealizations of modal sequents derived without the use of justificational rules have derivable corresponding formulas.

For the step of the main induction, consider the premise of the hth justificational rule. Its $(h-1)$-prerealization is derivable by IH. For each of the justificational rules we will show how to apply an additional substitution to make its conclusion derivable. By the Substitution Property (Lemma 3.7), this substitution preserves the derivability of all h-prerealizations of modal sequents whose $(h-1)$-prerealizations are derivable by the IH, including the premise of the hth justificational rule. Thus, the h-prerealization of its conclusion is also derivable and the argument about simple rules can be applied to extend this result down until the next justificational rule.

The cases of the $\mathsf{k}_\Box$ and $\mathsf{4}_\Box$ rules are treated the same way as in [10] by means of Corollary 3.4. Thus, it remains to process the two remaining justificational rules. We start with the case where the hth rule is

$$\mathsf{k}_\Diamond \frac{A_1^\circ, \ldots, A_r^\circ, C^\circ \Rightarrow D^\circ}{\Box_{k_1} A_1^\circ, \ldots, \Box_{k_r} A_r^\circ, B_1^\circ, \ldots, B_p^\circ, \Diamond_l C^\circ \Rightarrow \Diamond_j D^\circ}$$

and this introduction of $\Diamond_j$ is the qth among consequent introductions of $\Diamond_j$ in justificational rules. Assume we have a JL-derivation of $A_1 \wedge \ldots \wedge A_n \wedge C \supset D$, the $h-1$-prerealization of the premise of the rule. By Corollary 3.5 there is a witness term $\mu(x_1, \ldots, x_r, \beta)$ such that

$$\vdash_{\mathsf{JL}} z_{k_1} : A_1 \wedge \ldots \wedge z_{k_r} : A_r \wedge \omega_l : C \supset \mu(z_{k_1}, \ldots, z_{k_r}, \omega_l) : D.$$

We define $\sigma_h\colon \beta_{j,q} \mapsto \mu(z_{k_1},\ldots,z_{k_r},\omega_l)$. Note that σ_h affects exactly one witness variable, which is neither $y_{i,0}$ nor $\beta_{i,0}$ and which corresponds to the justificational rule being processed. In particular, $\beta_{j,q} = \beta_{j,q}\tau_{h-1}$ and $\beta_{j,q}\sigma_h = \beta_{j,q}\tau_h$. Thus,

$$\vdash_{\mathsf{JL}} z_{k_1}:A_1 \wedge \ldots \wedge z_{k_r}:A_r \wedge \omega_l : C \supset \beta_{j,q}\tau_h : D.$$

Applying the Substitution Property, we obtain

$$\vdash_{\mathsf{JL}} z_{k_1}:(A_1\sigma_h) \wedge \ldots \wedge z_{k_r}:(A_r\sigma_h) \wedge \omega_l : (C\sigma_h) \supset \beta_{j,q}\tau_h : (D\sigma_h)$$

because (a) σ_h does not affect any proof variable, including $z_{k_1},\ldots,z_{k_r}$, (b) σ_h does not affect the witness variable ω_l, which must be different from β_{j_k} because $j \neq l$ as indices of diamonds of opposite polarity, and (c) σ_h does not affect $\beta_{j,q}\tau_h = \mu(z_{k_1},\ldots,z_{k_r},\omega_l)$ because the only variables occurring in it are $z_{k_1},\ldots,z_{k_r},\omega_l$. It follows that

$$\vdash_{\mathsf{JL}} z_{k_1}:(A_1\sigma_h) \wedge \ldots \wedge z_{k_r}:(A_r\sigma_h) \wedge B_1\sigma_h \wedge \ldots \wedge B_p\sigma_h \wedge \omega_l : (C\sigma_h) \supset \omega_j\tau_h : (D\sigma_h)$$

where $\omega_j = \beta_{j,1} \sqcup \ldots \sqcup \beta_{j,q} \sqcup \ldots \sqcup \beta_{j,m_j}$.

The case for the rule

$$\mathsf{d}\ \frac{A_1^\circ,\ldots,A_r^\circ \Rightarrow D^\circ}{\Box_{k_1}A_1^\circ,\ldots,\Box_{k_r}A_r^\circ,B_1^\circ,\ldots,B_p^\circ,\Rightarrow \Diamond_j D^\circ}$$

is similar except we use $C = \top$ based on the IH that $\vdash_{\mathsf{JL}} A_1 \wedge \ldots \wedge A_n \supset D$. Repeating all the steps for $\mathsf{k}_\Diamond$ and using a fresh variable β for $\Diamond\top$, we obtain

$$\vdash_{\mathsf{JL}} z_{k_1}:(A_1\sigma_h) \wedge \ldots \wedge z_{k_r}:(A_r\sigma_h) \wedge B_1\sigma_h \wedge \ldots \wedge B_p\sigma_h \wedge \beta : (\top\sigma_h) \supset \omega_j\tau_h : (D\sigma_h)$$

Since $\top\sigma_h = \top$ and $\vdash_{\mathsf{JL}} \beta : \top$ it follows that

$$\vdash_{\mathsf{JL}} z_{k_1}:(A_1\sigma_h) \wedge \ldots \wedge z_{k_r}:(A_r\sigma_h) \wedge B_1\sigma_h \wedge \ldots \wedge B_p\sigma_h \supset \omega_j\tau_h : (D\sigma_h). \qquad \Box$$

The crucial difference between justificational and simple rules is that, unlike the former, the latter require an additional substitution on top of all the previous ones.

5 Conclusion and future works

In this paper, we proposed justification counterparts for some constructive modal logics, which, for the first time, employ the notion of witness terms to realize the $\Diamond$-modality. This led us to define an operator combining proof terms and witness terms, which is crucial to the realization of the constructive modal axiom k_2. However, surprisingly, the only other operation needed on witness terms is the disjoint union, an equivalent to the sum for proof terms. In particular, while the $\Box$-version of the 4-axiom traditionally requires the proof checker operator !, the $\Diamond$-version of axiom 4 do not seem to necessitate any additional operation on witness terms. In the following, we list a handful of directions for future work.

- We have not investigated in detail the semantics of the logics we proposed. It seems that modular models from [6] would provide a good basis, but require significant adjustments.

$$\mathsf{k4}_{\Box}\,\frac{\Box\Gamma,\Gamma\Rightarrow A}{\Delta,\Box\Gamma,\Rightarrow\Box A}\qquad \mathsf{k4}_{\Diamond}\,\frac{\Box\Gamma,\Gamma,B\Rightarrow A}{\Delta,\Box\Gamma,\Diamond B\Rightarrow\Diamond A}\qquad \mathsf{k4}'_{\Diamond}\,\frac{\Box\Gamma,\Gamma,B\Rightarrow\Diamond A}{\Delta,\Box\Gamma,\Diamond B\Rightarrow\Diamond A}$$

Figure 6: More rules for modalities

- We have chosen to work with the logics that have simple cut-free sequent calculi, a property on which the realization proof strongly relies. The same method can be further extended to CK4 and CD4 that are obtained from CK and CD, respectively, by adding the 4-axiom. The corresponding sequent systems are obtained via the rules in Fig. 6:

$$\begin{array}{rcl}\mathsf{LCK4} & = & \mathsf{G3ip}+\mathsf{k4}_{\Box}+\mathsf{k4}_{\Diamond}+\mathsf{k4}'_{\Diamond}\\ \mathsf{LCD4} & = & \mathsf{G3ip}+\mathsf{k4}_{\Box}+\mathsf{k4}_{\Diamond}+\mathsf{k4}'_{\Diamond}+\mathsf{d}\end{array}\tag{4}$$

  We decided to forego this extention for pragmatic reasons: without a cut-free calculi for these constructive modal logics in the literature we would need to provide a full cut-elimination proof. Even though this is a straightforward exercise adapting for example the proof from [18], it would have changed the focus of this paper.

- We believe that our way of justifying the $\Diamond$-modality would similarly work for the "intuitionistic variant" of modal logic [21], which is obtained from the constructive variant by adding the axioms

$$\mathsf{k}_3\colon \Diamond(A\vee B)\supset(\Diamond A\vee\Diamond B)\qquad \mathsf{k}_4\colon (\Diamond A\supset\Box B)\supset\Box(A\supset B)\qquad \mathsf{k}_5\colon \Diamond\bot\supset\bot\tag{5}$$

  There are no ordinary sequent calculi for such logics, so the proof of realization provided here could not be straightforwardly adapted. However, there are nested sequent calculi for all logics in the *intuitionistic S5-cube* [23], even in a focused variant [11], which means that we might still be able to prove a realization theorem by extending the method used in [17].

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References

[1] R. Arisaka, A. Das & L. Straßburger (2015): *On Nested Sequents for Constructive Modal Logic*. *LMCS* 11(3:7).

[2] S.N. Artemov (1995): *Operational modal logic*. Technical Report MSI 95–29, Cornell University.

[3] S.N. Artemov (2001): *Explicit Provability and Constructive Semantics*. *Bulletin of Symbolic Logic* 7(1), pp. 1–36.

[4] S.N. Artemov (2002): *Unified Semantics for Modality and λ-terms via Proof Polynomials*. In K. Vermeulen & A. Copestake, editors: *Algebras, Diagrams and Decisions in Language, Logic and Computation*, *CSLI Lecture Notes* 144, CSLI Publications, pp. 89–118.

[5] S.N. Artemov (2008): *The Logic of Justification*. *Review of Symbolic Logic* 1(4), pp. 477–513.

[6] S.N. Artemov (2012): *The Ontology of Justifications in the Logical Setting*. *Studia Logica* 100(1–2), pp. 17–30.

[7] S.N. Artemov & R. Iemhoff (2007): *The Basic Intuitionistic Logic of Proofs*. Journal of Symbolic Logic 72(2), pp. 439–451.

[8] S.N. Artemov, E.L. Kazakov & D. Shapiro (1999): *Logic of knowledge with justifications*. Technical Report CFIS 99–12, Cornell University.

[9] G.M. Bierman & V. de Paiva (2000): *On an Intuitionistic Modal Logic*. Studia Logica 65(3), pp. 383–416.

[10] V.N. Brezhnev (2000): *On explicit counterparts of modal logics*. Technical Report CFIS 2000–05, Cornell University.

[11] K. Chaudhuri, S. Marin & L. Straßburger (2016): *Modular Focused Proof Systems for Intuitionistic Modal Logics*. In D. Kesner & B. Pientka, editors: 1st International Conference on Formal Structures for Computation and Deduction, FSCD 2016, June 22-26, 2016, Porto, Portugal, LIPIcs 52, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, pp. 16:1–16:18.

[12] E. Dashkov (2008): *Intuitionistic Logic of Proofs*. Preprint 269, Logic Group Preprint Series, Department of Philosophy, Utrecht University.

[13] M. Fitting (2005): *The logic of proofs, semantically*. Annals of Pure and Applied Logic 132(1), pp. 1–25.

[14] M. Fitting (2016): *Modal Logics, Justification Logics, and Realization*. Annals of Pure and Applied Logic 167(8), pp. 615–648.

[15] K. Gödel (1933): *Eine Interpretation des intuitionistischen Aussagenkalküls*. Ergebnisse eines mathematischen Kolloquiums 4, pp. 39–40.

[16] K. Gödel (1995): *Vortrag bei Zilsel/Lecture at Zilsel's (*1938a)*. In S. Feferman, J.W. Dawson, Jr., W. Goldfarb, C. Parsons & R.M. Solovay, editors: Unpublished essays and lectures, Kurt Gödel Collected Works III, Oxford University Press, pp. 86–113.

[17] R. Goetschi & R. Kuznets (2012): *Realization for justification logics via nested sequents: Modularity through embedding*. Annals of Pure and Applied Logic 163(9), pp. 1271–1298.

[18] B. Lellmann & D. Pattinson (2013): *Constructing cut free sequent systems with ontext restrictions based on classical or intuitionistic logic*. ICLA 2013 7750, pp. 148–160.

[19] M. Marti & T. Studer (2016): *Intuitionistic Modal Logic made Explicit*. IfCoLog Journal of Logics and their Applications 3(5), pp. 877–901.

[20] M. Mendler & S. Scheele (2011): *Cut-free Gentzen calculus for multimodal CK*. Inf. Comput. 209(12), pp. 1465–1490.

[21] A. Simpson (1994): *The Proof Theory and Semantics of Intuitionistic Modal Logic*. Ph.D. thesis, University of Edinburgh.

[22] G. Steren & E. Bonelli (2014): *Intuitionistic Hypothetical Logic of Proofs*. In V. de Paiva, M. Benevides, V. Nigam & E. Pimentel, editors: Proceedings of the 6th Workshop on Intuitionistic Modal Logic and Applications (IMLA 2013) in association with UNILOG 2013, Rio de Janeiro, Brazil, 7 April 2013, Electronic Notes in Theoretical Computer Science 300, Elsevier, pp. 89–103.

[23] L. Straßburger (2013): *Cut Elimination in Nested Sequents for Intuitionistic Modal Logics*. In F. Pfenning, editor: FoSSaCS'13, LNCS 7794, Springer, pp. 209–224.

[24] A.S. Troelstra & H. Schwichtenberg (2000): *Basic Proof Theory*, second edition. Cambridge University Press.

[25] H. Wansing (2002): *Sequent systems for modal logics*, second edition, pp. 61–145. 8, Kluwer.

[26] D. Wijesekera (1990): *Constructive Modal Logics I*. Ann. Pure Appl. Logic 50(3), pp. 271–301.